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Chebychev Coefficients in Approximation of Powers of x

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Let $E_n(\alpha) = \min_{c_i} \max_{0 \leqslant x \leqslant 1} |x^\alpha - (c_0 + c_1 x + \cdots + c_n x^n)|$ and $E'_n(\alpha) = \min_{c_i} \max_{0 \leqslant x \leqslant 1} |x^\alpha - (c_1 x + c_2 x^2 + \cdots + c_n x^n)|$, where $n \in N$, $\alpha > 0$, and c_i is real for each i. Also denote the coefficient of x^u in the Chebychev polynomial T_v by $\operatorname{coef}(u, v)$.

THEOREM. *Let $n > 0$ and k be integers so that $0 \leqslant k \leqslant n$ and α a real number.*

(a) *If $\alpha \in [k, k+1]$, then $E_n(\alpha) < E'_n(\alpha) < 1/|\operatorname{coef}(2k, 2n)|$.*

(b) *If $\alpha \in [k+\frac{1}{2}, k+1]$, then $E_n(\alpha) < E'_n(\alpha) < 1/|\operatorname{coef}(2k+1, 2n+1)|$.*

1. INTRODUCTION

Bell and Shah have used oscillating generalized polynomials [2–4] to find the best uniformly approximating polynomial of degree n on [0, 1] to functions of the form $f(x) = x^r$, where r is a positive rational number. They then determined lower bounds for

$$E_n(r) = \min_{c_i} \max_{0 \leqslant x \leqslant 1} |x^r - (c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n)|.$$

This work was motivated by Bernstein's results [5] on the approximation of $|x|$ on $[-1, 1]$, which is equivalent to having $r = \frac{1}{2}$ and approximating on [0, 1].

In this paper we study the functions

$$E_n(\alpha) = \min_{c_i} \max_{0 \leqslant x \leqslant 1} |x^\alpha - (c_0 + c_1 x + \cdots + c_n x^n)|$$

and

$$E'_n(\alpha) = \min_{c_i} \max_{0 \leqslant x \leqslant 1} |x^\alpha - (c_1 x + c_2 x^2 + \cdots + c_n x^n)|,$$

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where $n \in N$, $\alpha > 0$, and c_i is real for each i. In so doing, the properties of Chebychev polynomials and of oscillating generalized polynomials are extremely useful in finding upper bounds for $E_n'(\alpha)$ and $E_n(\alpha)$ for many α's. The technique that is employed utilizes every coefficient in every Chebychev polynomial.

2. Oscillating Generalized Polynomials

Let $0 \leqslant \alpha(0) < \alpha(1) < \cdots < \alpha(n)$ be given rational numbers. Then $p(x) = c_0 x^{\alpha(0)} + c_1 x^{\alpha(1)} + \cdots + c_n x^{\alpha(n)}$, where the c_i's are real, is said to be a generalized polynomial (gp). If $\max_{0 \leqslant x \leqslant 1} |p(x)|$ is attained for exactly $n+1$ values of x in $[0, 1]$, then $p(x)$ is said to be an oscillating generalized polynomial (ogp) in $[0, 1]$.

The following facts about gps and ogps are stated: (i)–(vi) [2], (vii) [6], and (viii)–(xi) [9].

(i) (Property D).

(a) For every set of nonzero real numbers $\{c_0, c_1, ..., c_n\}$ and every set of rational numbers $\{\alpha(0), \alpha(1), ..., \alpha(n)\}$ with $0 \leqslant \alpha(0) < \alpha(1) < \cdots < \alpha(n)$, the number of zeros, a zero of order k counted as k zeros, in $(0, 1]$ of the generalized polynomial

$$p(x) = c_0 x^{\alpha(0)} + c_1 x^{\alpha(1)} + \cdots + c_n x^{\alpha(n)}$$

is at most equal to the number of variations of sign in the sequence $\{c_0, c_1, ..., c_n\}$.

(b) With the sets $\{c_0, c_1, ..., c_n\}$ and $\{\alpha(0), \alpha(1), ..., \alpha(n)\}$ as in (a), the number of zeros, a zero of order k counted as k zeros, in $(0, 1]$ of $p'(x)$ is at most equal to the number of variations of sign in the sequence $\{c_0, c_1, ..., c_n\}$.

(ii) To a given finite set of nonnegative exponents, there corresponds an ogp in $[0, 1]$ which is unique except for a constant factor.

(iii) Let $M = \max_{0 \leqslant x \leqslant 1} |p(x)|$. An ogp $p(x)$ assumes the values $\pm M$ alternately at $n+1$ points in $[0, 1]$.

(iv) Let $p(x) = \sum_{j=0}^{i-1} A_j x^{\alpha(j)} + x^m + \sum_{j=i+1}^{n} A_j x^{\alpha(j)}$ and $q(x) = \sum_{j=0}^{i-1} B_j x^{\beta(j)} + x^m + \sum_{j=i+1}^{n} B_j x^{\beta(j)}$ be ogps with $0 \leqslant \alpha(0) < \beta(0) < \cdots < \alpha(i-1) < \beta(i-1) < m < \beta(i+1) < \alpha(i+1) < \cdots < \beta(n) < \alpha(n)$. Then $\max_{0 \leqslant x \leqslant 1} |q(x)| < \max_{0 \leqslant x \leqslant 1} |p(x)|$.

(v) The coefficients of an ogp $p(x) = a_0 x^{\alpha(0)} + a_1 x^{\alpha(1)} + \cdots + a_n x^{\alpha(n)}$ alternate in sign.

(vi) Let $p(x) = \sum_{j=0}^{n} A_j x^{\alpha(j)}$ be an ogp in $[0, 1]$ and let $q(x) =$

$\sum_{j=0}^{n} B_j x^{\alpha(j)}$ (all B_j's real) be another generalized polynomial. Suppose $B_j = A_j$ for at least one j, where $\alpha(j) > 0$. Then $\max_{0 \leqslant x \leqslant 1} |q(x)| > \max_{0 \leqslant x \leqslant 1} |p(x)|$.

(vii) $E_n(\alpha) > E'_n(\alpha)/2$ for $\alpha > 0$ and rational.

(viii) Let $p(x) = a_0 x^{\alpha(0)} + \sum_{k=1}^{n} a_k x^{\alpha(k)}$ and $q(x) = a_0 x^{\alpha(0)} + \sum_{k=1}^{n} b_k x^{\beta(k)}$ be ogps such that $0 < \alpha(0) < \alpha(1) < \cdots < \alpha(n)$, $\alpha(0) < \beta(1) < \beta(2) < \cdots < \beta(n)$, and for $j = 1, ..., n$, $\alpha(j) < \beta(j)$. Then $\max_{0 \leqslant x \leqslant 1} |p(x)| < \max_{0 \leqslant x \leqslant 1} |q(x)|$.

(ix) For $\alpha \in [n, n+1]$, $E_n(\alpha) \leqslant 1/2^{2n+1}$ and for $\alpha \in (n+1, \infty)$, $E_n(\alpha) > 1/2^{2n+1}$.

(x) Each of E_n and E'_n is a continuous function on $(0, \infty)$.

(xi) Each of E_n and E'_n is strictly decreasing on $(0, 1]$ and strictly increasing on $[n, \infty)$.

3. Further Results on Oscillating Generalized Polynomials

THEOREM 1. *Let*

$$p(x) = a_0 x^{\alpha(0)} + \cdots + a_{i-1} x^{\alpha(i-1)} + x^m + a_{i+1} x^{\alpha(i+1)} + \cdots + a_n x^{\alpha(n)}$$

and

$$q(x) = b_0 x^{\beta(0)} + \cdots + b_{i-1} x^{\beta(i-1)} + x^m + b_{i+1} x^{\beta(i+1)} + \cdots + b_n x^{\beta(n)}$$

be the unique ogps with 1 *as the coefficient of* x^m *and the positive rational exponents* $\{\alpha(0), ..., \alpha(i-1), m, \alpha(i+1), ..., \alpha(n)\}$ *and* $\{\beta(0), ..., \beta(i-1), m, \beta(i+1), ..., \beta(n)\}$, *respectively, where* $0 < \alpha(0) < \alpha(1) < \cdots < \alpha(i-1) < m < \alpha(i+1) < \cdots < \alpha(n)$; $0 \leqslant \beta(0) < \beta(1) < \cdots < \beta(i-1) < m < \beta(i+1) < \cdots < \beta(n)$; *for* $j = 0, 1, ..., i-1$, $\beta(j) < \alpha(j)$, *and, for* $j = i+1, ..., n$, $\alpha(j) < \beta(j)$. *Then*

$$\max_{0 \leqslant x \leqslant 1} |p(x)| < \max_{0 \leqslant x \leqslant 1} |q(x)|.$$

Proof. The α's and the β's in this argument are all to be rational. First choose $\{\alpha(0,0), \alpha(1,0), ..., \alpha(i-1,0), \alpha(i+1,0), ..., \alpha(n,0)\}$ by $\beta(0) < \alpha(0,0) < \min\{\alpha(0), \beta(1)\}$; $\alpha(j,0) \in (\max\{\beta(j), \alpha(j-1)\}, \alpha(j))$, for $j = 1, 2, ..., i-1$; $\alpha(j,0) \in (\alpha(j), \min\{\alpha(j+1), \beta(j)\})$, for $j = i+1, ..., n-1$; and $\alpha(n,0) \in (\alpha(n), \beta(n))$.

Next suppose that $k \in N$ with $0 < k < i-2$ and that $\{\alpha(0,k), \alpha(1,k), ..., \alpha(i-1,k), \alpha(i+1,k), ..., \alpha(n,k)\}$ has been chosen so that $\beta(0) < \alpha(0,k) < \alpha(0,k-1) < \beta(1) < \alpha(1,k) < \alpha(1,k-1) < \beta(2) < \cdots < \beta(k) < \alpha(k,k) < \min\{\beta(k+1), \alpha(k,k-1)\}$; $\alpha(j,k) \in (\max\{\beta(j), \alpha(j-1,k-1)\},$

$\alpha(j, k-1))$, for $j=k+1, k+2, ..., i-1$; $\alpha(j, k) \in (\alpha(j, k-1), \min\{\alpha(j+1, k-1), \beta(j)\})$, for $j=i+1, ..., n-1$; and $\alpha(n, k) \in (\alpha(n, k-1), \beta(n))$.

Then choose $\{\alpha(0, k+1), \alpha(1, k+1), ..., \alpha(i-1, k+1), \alpha(i+1, k+1), ..., \alpha(n, k+1)\}$ so that $\beta(0) < \alpha(0, k+1) < \alpha(0, k) < \beta(1) < \alpha(1, k+1) < \alpha(1, k) < \beta(2) < \cdots < \beta(k+1) < \alpha(k+1, k+1) < \min\{\beta(k+2), \alpha(k+1, k)\}$; $\alpha(j, k+1) \in (\max\{\beta(j), \alpha(j-1, k)\}, \alpha(j, k))$, for $j=k+2, k+3, ..., i-1$; $\alpha(j, k+1) \in (\alpha(j, k), \min\{\alpha(j+1, k), \beta(j)\})$, for $j=i+1, ..., n-1$; and $\alpha(n, k+1) \in (\alpha(n, k), \beta(n))$.

Continue by choosing $\{\alpha(0, i+2), \alpha(1, i+2), ..., \alpha(i-1, i+2), \alpha(i+1, i+2), ..., \alpha(n, i+2)\}$ so that $\beta(0) < \alpha(0, i+2) < \alpha(0, i-2) < \beta(1) < \alpha(1, i+2) < \alpha(1, i-2) < \cdots < \beta(i-1) < \alpha(i-1, i+2) < \alpha(i-1, i-2)$; $\alpha(j, i+2) \in (\alpha(j, i-2), \min\{\alpha(j+1, i-2), \beta(j)\})$, for $j=i+1, ..., n-1$; and $\alpha(n, i+2) \in (\max\{\beta(n-1), \alpha(n, i-2)\}, \beta(n))$.

Next suppose for $2 < t < n-i$ that $\{\alpha(0, i+t), \alpha(1, i+t), ..., \alpha(i-1, i+t), \alpha(i+1, i+t), ..., \alpha(n, i+t)\}$ has been chosen such that $\beta(0) < \alpha(0, i+t) < \alpha(0, i+t-1) < \beta(1) < \alpha(1, i+t) < \alpha(1, i+t-1) < \beta(2) < \cdots < \beta(i-1) < \alpha(i-1, i+t) < \alpha(i-1, i+t-1)$; $\alpha(j, i+t) \in (\alpha(j, i+t-1), \min\{\alpha(j+1, i+t-1), \beta(j)\})$, for $j=i+1, ..., n-t+1$; $\alpha(j, i+t) \in (\max\{\beta(j-1), \alpha(j, i+t-1)\}, \min\{\alpha(j+1, i+t-1), \beta(j)\})$, for $j=n-t+2, ..., n-1$; and $\alpha(n, i+t) \in (\alpha(n, i+t-1), \beta(n))$.

Then choose $\{\alpha(0, i+t+1), \alpha(1, i+t+1), ..., \alpha(i-1, i+t+1), \alpha(i+1, i+t+1), ..., \alpha(n, i+t+1)\}$ so that $\beta(0) < \alpha(0, i+t+1) < \alpha(0, i+t) < \beta(1) < \alpha(1, i+t+1) < \alpha(1, i+t) < \beta(2) < \cdots < \beta(i-1) < \alpha(i-1, i+t+1) < \alpha(i-1, i+t)$; $\alpha(j, i+t+1) \in (\alpha(j, i+t), \min\{\alpha(j+1, i+t), \beta(j)\})$, for $j=i+1, ..., n-t$; $\alpha(j, i+t+1) \in (\max\{\beta(j-1), \alpha(j, i+t)\}, \min\{\alpha(j+1, i+t), \beta(j)\})$, for $j=n-t+1, ..., n-1$; and $\alpha(n, i+t+1) \in (\alpha(n, i+t), \beta(n))$.

Now for each $r=0, 1, 2, ..., i-2, i+2, ..., n$, define

$$p_r(x) = b_0^{(r)} x^{\alpha(0,r)} + b_1^{(r)} x^{\alpha(1,r)} + \cdots + b_{i-1}^{(r)} x^{\alpha(i-1,r)} + x^m + b_{i+1}^{(r)} x^{\alpha(i+1,r)} + \cdots + b_n^{(r)} x^{\alpha(n,r)}$$

to be the unique ogp with exponents $\{\alpha(0, r), \alpha(1, r), ..., \alpha(i-1, r), \alpha(i+1, r), ..., \alpha(n, r)\}$ and 1 the coefficients of x^m. Then by (iv) of Section 2,

$$\max_{0 \leqslant x \leqslant 1} |p(x)| < \max_{0 \leqslant x \leqslant 1} |p_0(x)| < \max_{0 \leqslant x \leqslant 1} |p_1(x)| < \cdots < \max_{0 \leqslant x \leqslant 1} |p_{i-2}(x)| < \max_{0 \leqslant x \leqslant 1} |p_{i+2}(x)| < \cdots < \max_{0 \leqslant x \leqslant 1} |p_n(x)| < \max_{0 \leqslant x \leqslant 1} |q(x)|.$$

LEMMA 2. *Let*

$$p(x) = a_0 x^{\alpha(0)} + a_1 x^{\alpha(1)} + \cdots + a_{n-1} x^{\alpha(n-1)} + x^{\alpha(n)}$$

and

$$q(x)=b_0x^{\beta(0)}+b_1x^{\beta(1)}+\cdots+b_{n-1}x^{\beta(n-1)}+x^{\alpha(n)}$$

be two ogps with $0\leqslant\alpha(0)<\beta(0)<\alpha(1)<\beta(1)<\cdots<\alpha(n-1)<\beta(n-1)<\alpha(n)$. *Then*

$$\max_{0<x<1}|q(x)|<\max_{0\leqslant x\leqslant 1}|p(x)|.$$

Proof. Note that

$$\begin{aligned}q(x)-p(x)=&-a_0x^{\alpha(0)}+b_0x^{\beta(0)}-a_1x^{\alpha(1)}\\&+b_1x^{\beta(1)}-\cdots-a_{n-1}x^{\alpha(n-1)}+b_{n-1}x^{\beta(n-1)}\end{aligned}$$

has n sign variations. Therefore by Property D, there are at most n positive zeros of $q(x)-p(x)$. Next suppose that $\max_{0\leqslant x\leqslant 1}|q(x)|\geqslant\max_{0\leqslant x\leqslant 1}|p(x)|$ and let $x_1, x_2, ..., x_{n+1}$ be the set of points in $[0, 1]$ at which $|q(x)|$ is maximum. Consequently there are n positive zeros $z_1, z_2, ..., z_n$ such that $z_0\leqslant x_1\leqslant z_1\leqslant x_2\leqslant z_2\leqslant x_3\leqslant\cdots\leqslant x_n\leqslant z_n\leqslant x_{n+1}\leqslant z_{n+1}$, where 0 is denoted by z_0 and 2 by z_{n+1}. Note that $q(x)-p(x)$ has the sign of b_{n-1} and is negative on (z_n, z_{n+1}) and in general $q(x)-p(x)$ has the sign $(-1)^i$ on (z_{n+1-i}, z_{n+2-i}) for $i=1, 2, ..., n+1$. Also it follows for some j that $z_{n+1-j}<x_{n+2-j}<z_{n+2-j}$. Therefore both $[q(x_{n+2-j})-p(x_{n+2-j})](-1)^{n+2-j}$ and $[q(x_{n+2-j})](-1)^{n+2-j}$ have the sign of $(-1)^{n+2}$.

On the other hand $q(x_{n+1})>0$, $q(x_n)<0$, and in general $q(x_{n+2-i})$ has the sign of $(-1)^{i+1}$. Therefore $[q(x_{n+2-j})](-1)^{n+2-j}$ has the sign of $(-1)^{n+3}$. This contradiction implies that $\max_{0\leqslant x\leqslant 1}|q(x)|<\max_{0\leqslant x\leqslant 1}|p(x)|$.

THEOREM 3. *Let*

$$p(x)=a_0x^{\alpha(0)}+a_1x^{\alpha(1)}+\cdots+a_{n-1}x^{\alpha(n-1)}+x^{\alpha(n)}$$

and

$$q(x)=b_0x^{\beta(0)}+b_1x^{\beta(1)}+\cdots+b_{n-1}x^{\beta(n-1)}+x^{\alpha(n)}$$

be two ogps with $0\leqslant\alpha(0)<\alpha(1)<\cdots<\alpha(n-1)<\alpha(n)$, $0<\beta(0)<\beta(1)<\cdots<\beta(n-1)<\alpha(n)$, *and* $\alpha(i)<\beta(i)$ *for each* $i=0, 1, 2, ..., n-1$. *Then*

$$\max_{0\leqslant x\leqslant 1}|q(x)|<\max_{0\leqslant x\leqslant 1}|p(x)|.$$

Proof. All of the α's and β's are assumed to be rational. First choose $\{\alpha(0,0), \alpha(1,0), ..., \alpha(n-1,0)\}$ so that for each $i=0, 1, ..., n-2$, $\alpha(i,0) \in (\alpha(i), \min\{\alpha(i+1), \beta(i)\})$ and $\alpha(n-1,0) \in (\max\{\alpha(n-1), \beta(n-2)\}, \beta(n-1))$.

Denote $\alpha(i)$ by $\alpha(i, -1)$ for each $i=0, 1, ..., n-1$. Next suppose that j is a member of $\{0, 1, 2, ..., n-3\}$ and $\{\alpha(0,j), \alpha(1,j), ..., \alpha(n-1,j)\}$ has been chosen so that for $i=0, 1, ..., n-j-2$, $\alpha(i,j) \in (\alpha(i,j-1), \min\{\alpha(i+1, j-1), \beta(i)\})$ and for $i=n-j-1, n-j, ..., n-1$, $\alpha(i,j) \in (\max\{\alpha(i,j-1), \beta(i-1)\}, \min\{\alpha(i+1,j-1), \beta(i)\})$. Then choose $\{\alpha(0,j+1), \alpha(1,j+1), ..., \alpha(n-1,j+1)\}$ so that for $i=0, 1, 2, ..., n-j-3$, $\alpha(i,j+1) \in (\alpha(i,j), \min\{\alpha(i+1,j), \beta(i)\})$ and for $i=n-j-2, n-j-1, ..., n-1$, $\alpha(i,j+1) \in (\max\{\alpha(i,j), \beta(i-1)\}, \min\{\alpha(i+1,j), \beta(i)\})$.

Next for $r=0, 1, 2, ..., n-2$, let $p_r = a_0^{(r)} x^{\alpha(0,r)} + a_1^{(r)} x^{\alpha(1,r)} + \cdots + a_{n-1}^{(r)} x^{\alpha(n-1,r)} + x^{\alpha(n)}$ be the unique ogp with exponents $\{\alpha(0,r), \alpha(1,r), ..., \alpha(n-1,r), \alpha(n)\}$ and 1 as the coefficient of $x^{\alpha(n)}$. Then by Lemma 2,

$$\max_{0\leqslant x\leqslant 1} |p(x)| > \max_{0\leqslant x\leqslant 1} |p_0(x)| > \max_{0\leqslant x\leqslant 1} |p_1(x)| > \cdots > \max_{0\leqslant x\leqslant 1} |p_{n-2}(x)| > \max_{0\leqslant x\leqslant 1} |q(x)|.$$

4. Applications of Oscillating Generalized Polynomials

In the following denote the coefficient of x^μ in the Chebychev polynomial T_v by coef(μ, v).

Lemma 4. *Let n be a positive integer and p/q be a rational number.*

(a) *If $p/q \in (0,1)$, then $E_n(p/q) < E_n'(p/q) < 1/|\text{coef}(0, 2n)|$.*

(b) *If $p/q \in (\frac{1}{2}, 1)$, then $E_n(p/q) < E_n'(p/q) < 1/|\text{coef}(1, 2n+1)|$.*

Proof. (a) This is trivial since $|\text{coef}(0, 2n)| = 1$.

(b) Actually the inequalities hold for $p/q \in (n/(2n+1), 1)$. Let $x^{p/q} + c_1 x + c_2 x^2 + \cdots + c_n x^n$ be the unique ogp with exponents $\{p/q, 1, 2, ..., n\}$ and 1 as the coefficient of $x^{p/q}$. Then

$$p(x) = x^p + c_1 x^q + c_2 x^{2q} + \cdots + c_n x^{nq}$$

and

$$\frac{T_{2n+1}(x^p)}{\text{coef}(1, 2n+1)} = x^p + a_3 x^{3p} + a_5 x^{5p} + \cdots + a_{2n+1} x^{(2n+1)p}$$

are both ogps. Since $q < 3p$, $2q < 5p$, ..., $nq < (2n+1)p$, by (viii),

$$E_n\left(\frac{p}{q}\right) < E'_n\left(\frac{p}{q}\right) = \max_{0 \leqslant x \leqslant 1} |p(x)|$$

$$< \max_{0 \leqslant x \leqslant 1} \left|\frac{T_{2n+1}(x^p)}{\operatorname{coef}(1, 2n+1)}\right| = \frac{1}{|\operatorname{coef}(1, 2n+1)|}.$$

LEMMA 5. *Let n and k be positive integers with $k < n$ and p/q be a rational number.*

(a) *If $p/q \in (k, k+1)$, then $E_n(p/q) < E'_n(p/q) < 1/|\operatorname{coef}(2k, 2n)|$.*

(b) *If $p/q \in (k+\frac{1}{2}, k+1)$, then $E_n(p/q) < E'_n(p/q) < 1/|\operatorname{coef}(2k+1, 2n+1)|$.*

Proof. In both (a) and (b) let $x^{p/q} + c_1 x + c_2 x^2 + \cdots + c_n x^n$ be the unique ogp with exponents $\{p/q, 1, 2, ..., n\}$ and 1 as the coefficient of $x^{p/q}$.

(a) Then

$$p(x) = c_1 x^q + c_2 x^{2q} + \cdots + c_k x^{kq} + x^p + c_{k+1} x^{(k+1)q} + \cdots + c_n x^{nq}$$

and

$$\frac{T_{2n}(x^{p/2k})}{\operatorname{coef}(2k, 2n)} = a_0 + a_2 x^{p/k} + a_4 x^{2p/k} + \cdots + a_{2(k-1)} x^{(k-1)p/k}$$
$$+ x^p + a_{2(k+1)} x^{(k+1)p/k} + \cdots + a_{2n} x^{np/k}$$

are both ogps. Since $0 < q$, $p/k < 2q$, ..., $((k-1)/k)p < kq$, and $(k+1)q < ((k+1)/k)p$, ..., $nq < np/k$, by Theorem 1

$$E_n\left(\frac{p}{q}\right) < E'_n\left(\frac{p}{q}\right) = \max_{0 \leqslant x \leqslant 1} |p(x)|$$

$$< \max_{0 \leqslant x \leqslant 1} \left|\frac{T_{2n}(x^{p/2k})}{\operatorname{coef}(2k, 2n)}\right| = \frac{1}{|\operatorname{coef}(2k, 2n)|}.$$

(b) Note that

$$\frac{T_{2n+1}(x^{p/(2k+1)})}{\operatorname{coef}(2k+1, 2n+1)}$$
$$= a_1 x^{p/(2k+1)} + a_3 x^{3p/(2k+1)} + \cdots + a_{2k-1} x^{(2k-1)p/(2k+1)} + x^p$$
$$+ a_{2k+3} x^{(2k+3)p/(2k+1)} + \cdots + a_{2n+1} x^{(2n+1)p/(2k+1)}$$

is also an ogp. Since $p/(2k+1)<q$, $3p/(2k+1)<2q, ..., ((2k-1)/(2k+1))p<kq$ and $(k+1)q<((2k+3)/(2k+1))p$, $(k+2)q<((2k+5)/(2k+1))p, ..., nq<((2n+1)/(2k+1))p$, by Theorem 1,

$$E_n\left(\frac{p}{q}\right)<E_n'\left(\frac{p}{q}\right)=\max_{0\leqslant x\leqslant 1}|p(x)|$$

$$<\max_{0\leqslant x\leqslant 1}\left|\frac{T_{2n+1}(x^{p/(2k+1)})}{\operatorname{coef}(2k+1,2n+1)}\right|=\frac{1}{|\operatorname{coef}(2k+1,2n+1)|}.$$

LEMMA 6. *Let n be a positive integer and p/q a rational number.*

(a) *If $p/q\in(n,n+1)$, then $E_n(p/q)<E_n'(p/q)<1/|\operatorname{coef}(2n,2n)|$.*

(b) *If $p/q\in(n+\frac{1}{2},n+1)$, then $E_n(p/q)<E_n'(p/q)<1/|\operatorname{coef}(2n+1,2n+1)|$.*

Proof. Let $x^{p/q}+c_1x+c_2x^2+\cdots+c_nx^n$ be as in Lemma 5.

(a) Then

$$p(x)=c_1x^q+c_2x^{2q}+\cdots+c_nx^{nq}+x^p$$

and

$$\frac{T_{2n}(x^{p/2n})}{\operatorname{coef}(2n,2n)}=a_0+a_2x^{p/n}+a_4x^{2p/n}+\cdots+a_{2(n-1)}x^{(n-1)p/n}+x^p$$

are both ogps. Since $0<q$, $p/n<2q, ..., ((n-1)/n)p<nq$, by Theorem 3,

$$E_n\left(\frac{p}{q}\right)<E_n'\left(\frac{p}{q}\right)=\max_{0\leqslant x\leqslant 1}|p(x)|$$

$$<\max_{0\leqslant x\leqslant 1}\left|\frac{T_{2n}(x^{p/2n})}{\operatorname{coef}(2n,2n)}\right|=\frac{1}{|\operatorname{coef}(2n,2n)|}.$$

(b) Note that

$$\frac{T_{2n+1}(x^{p/(2n+1)})}{\operatorname{coef}(2n+1,2n+1)}=a_1x^{p/(2n+1)}+a_3x^{3p/(2n+1)}$$

$$+\cdots+a_{2n-1}x^{(2n-1)p/(2n+1)}+x^p$$

is also an ogp. Since $p/(2n+1)<q$, $3p/(2n+1)<2q, ..., ((2n-1)/(2n+1))p<nq$, by Theorem 3,

$$E_n\left(\frac{p}{q}\right) < E'_n\left(\frac{p}{q}\right) = \max_{0 \leqslant x \leqslant 1} |p(x)|$$

$$< \max_{0 \leqslant x \leqslant 1} \left| \frac{T_{2n+1}(x^{p/(2n+1)})}{\operatorname{coef}(2n+1, 2n+1)} \right| = \frac{1}{|\operatorname{coef}(2n+1, 2n+1)|}.$$

By (ix), (x), and (xi) and Lemmas 4, 5, and 6, the following holds:

THEOREM 7. *Let $n > 0$ and k be integers so that $0 \leqslant k \leqslant n$ and α a real number.*

(a) *If $\alpha \in [k, k+1]$, then $E_n(\alpha) < E'_n(\alpha) < 1/|\operatorname{coef}(2k, 2n)|$.*

(b) *If $\alpha \in [k+\frac{1}{2}, k+1]$, then $E_n(\alpha) < E'_n(\alpha) < 1/|\operatorname{coef}(2k+1, 2n+1)|$.*

EXAMPLE 8. What is an upper bound to

$$E_5(\tfrac{11}{3}) = \max_{0 \leqslant x \leqslant 1} |x^{11/3} - (c_0 + c_1 x + \cdots + c_5 x^5)|,$$

where $c_0 + c_1 x + \cdots + c_5 x^5$ is the best approximation polynomial of degree 5 in the uniform norm to $x^{11/3}$ on $[0, 1]$?

Since $\frac{11}{3} \in (3, 4)$,

$$T_{10}(x) = -1 + 50x^2 - 400x^4 + 1120x^6 - 1280x^8 + 512x^{10},$$

and

$$T_{11}(x) = -11x + 220x^3 - 1232x^5 + 2816x^7 - 2816x^9 + 1024x^{11},$$

it follows that

$$\text{(a)} \quad E_5\left(\frac{11}{3}\right) < \frac{1}{|\operatorname{coef}(6, 10)|} = \frac{1}{|1120|} = \frac{1}{1120}$$

and

$$\text{(b)} \quad E_5\left(\frac{11}{3}\right) < \frac{1}{|\operatorname{coef}(7, 11)|} = \frac{1}{|2816|} = \frac{1}{2816}.$$

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